Both Halves of the Boundary

the reverse, the derivative, and the clock a reduction calculus already contains



⊢ ⊣

July 2026 · living document, revision 1

Abstract. A confluent reduction calculus normalizes: many terms flow to one normal form, and the flow is irreversible by construction. Its lattice of observations is nevertheless a *bi-Heyting* algebra, and a bi-Heyting algebra carries two negations, not one. The calculus is built on the first — the double Heyting negation $\mathsf{R} = \neg\neg$, whose fixed points are the normal forms. The second, the double co-Heyting supplement $\mathsf{k} = {\sim}{\sim}$, sits inside the algebra unused. We promote it. Once k is a first-class operator, every configuration splits as $a = \mathsf{k}a \vee \partial a$: its reversible interior core, joined with its boundary $\partial a = a \wedge {\sim}a$. That single identity is three things at once. It is a *reverse*: k is the canonical best-inverse to normalization, and ∂ measures exactly what the forward flow destroyed — a Landauer ledger in which erasing the boundary is the only irreversible act. It is a *derivative*: ∂ is a co-Heyting derivation, and it distributes over the parallel-composition tensor, so the boundary defect of a composite is the sum of its parts' — the linear core of a chain rule on reduction. And it is a *clock*: ∂ is the infinitesimal generator whose fixed locus is exactly the calculus's eigenforms, its normal forms, its ground state. We identify the objects on a three-valued re-entry classifier and again on the four-configuration lattice of an actual interaction net, and we resolve where the program stops: the boundary is a derivation but not a differential because it is an idempotent *projector* ($\partial^2 = \partial$). Its invariant is therefore K-theoretic — the rank ($1, 2, 3$ on the corpus) — and the genuine homology is concentrated in degree zero (the K-theory), with the reduction complex acyclic above it. The infinite periodic-cyclic tower degenerates onto that degree-zero invariant because an idempotent admits no infinitesimal deformation (§7). The results are checked in Lean 4 and, on the net-configuration lattice, independently in Cubical Agda; see §8. This paper is a living document: §10 names the phases still to land.

1 The operator a calculus forgets it has

A rewriting system that is confluent has a peculiar poverty. It computes a normal form and stops. The map from a term to its normal form is idempotent, order-preserving, and inflationary in the observational order — it is, in the exact technical sense, a *closure operator*. Closure operators are the mathematics of *forgetting the difference between a thing and its completion*: once you have closed, closing again does nothing. This is what makes a confluent calculus reliable, and it is also what makes it flat. There is no room, in a closure, for a derivative or a reverse: both would have to remember what the closure threw away.

The lattice a reduction calculus actually lives in is richer than the closure alone can see. Order the observations of a net by refinement and you get a distributive lattice; a finite distributive lattice is automatically *bi-Heyting*. A bi-Heyting algebra is a Heyting algebra — intuitionistic implication $\Rightarrow$, negation $\neg$, the logic of open sets — that is *also* co-Heyting: it carries a subtraction $\smallsetminus$ and a co-negation, the supplement $\sim(\cdot)$, the logic of closed sets. The two negations are genuinely different. Where the double negation $\neg\neg$ is a closure (it inflates: $a \le \neg\neg a$), the double supplement $\sim\sim$ is an *interior* (it deflates: $\sim\sim a \le a$). One completes; the other core-s.

The question. A confluent calculus wires in the closure $\mathsf{R} = \neg\neg$ — normalization *is* the double-negation nucleus, and its fixed points are exactly the normal forms. What happens if we also wire in the other one?

The claim. The interior co-nucleus $\mathsf{k} = \sim\sim$ is not decoration. Promoting it to a first-class *co-reduction* makes three operations available that a closure-only calculus provably cannot have, and all three are the same operator wearing three hats. *First*, a canonical reverse: k is the best inverse to normalization, and the gap between a and its interior — the boundary $\partial a = a \wedge \sim a$ — is exactly the information the forward reduction destroyed. Reduction becomes reversible modulo its boundary, with the irreversibility measured, not waved at. *Second*, a derivative: ∂ obeys a Leibniz law, and it distributes additively over the tensor that composes computations in parallel, which is the linear part of a chain rule. *Third*, a clock: ∂ is a derivation, hence the generator of a flow, and the flow's fixed points are precisely the calculus's normal forms — its stable self-referential configurations, its ground state.

The meaning. A boundary is the one place where an object and its complement touch. That a single operator built from the "closed-set" negation should govern a calculus's reverse, its derivative, and its clock at once is a statement about where irreversibility, differentiation, and time actually come from in a computational world: not from three mechanisms bolted on, but from the one seam the closure was busy forgetting. We do not prove that reading; we prove the identity $a = \mathsf{k}a \vee \partial a$ and its three consequences, and let the reading follow. The cohomology reframe in §7 then shows that the boundary itself is an idempotent projector: its finite invariant is K-theoretic rank, and the genuine differential is the reduction-coupled operator $d = \mathrm{step} \circ \partial$.

2 Two negations, and the operator between them

Ask what the smallest interesting carrier of this structure is. It is not Boolean — in a Boolean algebra $a \wedge \neg a = \bot$ always, so the boundary vanishes identically and there is nothing to promote. The smallest carrier with a nontrivial boundary is the three-element chain $\bot < \mathsf{s} < \top$, the re-entry classifier: two classical poles and one middle value s that is its own re-entry, neither true nor false

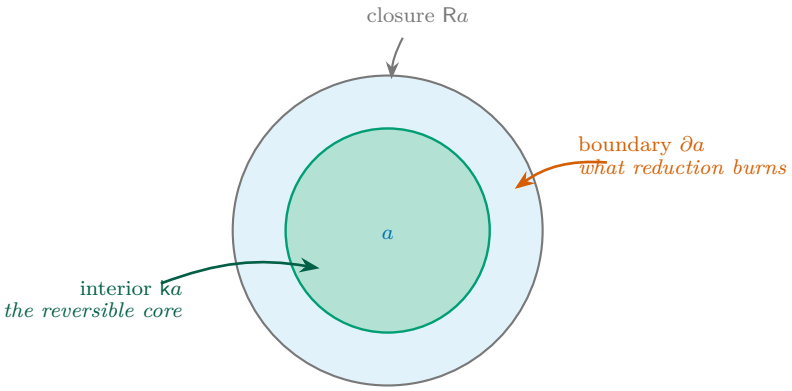


Figure 1: **Every configuration is its interior, its closure, and the ring between them.** The closure $\mathsf{R}a = \neg\neg a$ completes a outward; the interior $\mathsf{k}a = {\sim}{\sim}a$ cores it inward; the boundary $\partial a = a \wedge {\sim}a$ is the seam where they disagree. A confluent calculus uses only the outer completion. The whole of this paper is the claim that the inner core and the seam were there all along, and that $a = \mathsf{k}a \vee \partial a$ (Theorem 3.1) recovers a from them. Legible with no domain knowledge: a disk, its inside, and its edge.

but the mark that re-enters the form it draws. We take it as our first laboratory; §6 moves to an actual interaction net.

Definition 2.1 (The two negations and the boundary)**. Picture.** Two ways to say "not": one that opens ($\neg$, true only at the classical poles), one that closes (${\sim}(\cdot)$, its order-reversing dual). Their disagreement is the boundary. **Construction.** On the chain $\Omega_3 = \{\bot, \mathsf{s}, \top\}$ with meet $\wedge$ and join $\vee$, the Heyting negation is $\neg a = (a \Rightarrow \bot)$, giving $\neg\bot = \top$, $\neg\mathsf{s} = \bot$, $\neg\top = \bot$; the co-Heyting supplement is ${\sim}a$, the least c with $a \vee c = \top$, giving ${\sim}\bot = \top$, ${\sim}\mathsf{s} = \top$, ${\sim}\top = \bot$. The *closure* is $\mathsf{R} = \neg\neg$, the *interior* is $\mathsf{k} = {\sim}{\sim}$, and the *boundary* is $\partial a = a \wedge {\sim}a$. **Justification.** The definitions are forced by the two adjunctions, not chosen: $\Rightarrow$ is the right adjoint of $\wedge$ (Heyting), $\smallsetminus$ the left adjoint of $\vee$ (co-Heyting). Substitute the "nearby" operator — use $\neg\neg$ where ${\sim}{\sim}$ belongs — and the decomposition of Theorem 3.1 fails at the middle value; we exhibit that failure there. The supplement is the closed-set negation precisely because it is the subtraction from the top, and it is the subtraction from the top that makes the boundary a derivation (Theorem 4.1).

The boundary already carries the whole non-classicality of the carrier, and it does so as a fixed point: $\partial\mathsf{s} = \mathsf{s} \wedge {\sim}\mathsf{s} = \mathsf{s} \wedge \top = \mathsf{s}$. The middle value is where the boundary lives, and it is the only place: $\partial\bot = \bot$ and $\partial\top = \bot$. This is the sense in which the boundary is the obstruction to being classical — it vanishes exactly at the poles.

Proposition 2.2 (The boundary is the obstruction to classicality)**.** *$\partial a = \bot$ if and only if $\mathsf{k}a = \mathsf{R}a$ — the interior meets the closure — if and only if a is one of the classical poles $\{\bot, \top\}$. Equivalently, $\partial a = \bot \iff \mathsf{R}a = a$: the boundary vanishes exactly at the fixed points of the closure, which are exactly the normal forms.*

This is the hinge. The fixed points of the closure — the reduction's normal forms — are precisely the boundary's zero locus. Everything the calculus calls "done" is everything the boundary calls "classical." The middle value s is the single configuration that is neither, and it is where all three of our operations will draw their content.

3 The reverse: a fundamental theorem, and a ledger

Normalization is irreversible: it is a closure, and a closure is not injective. There is no inverse. But there is a canonical *best* inverse, and the algebra already knows it.

Theorem 3.1 (The fundamental theorem of the dual nucleus)**.** ***Picture.*** *Take any configuration apart into the part that survives reduction losslessly and the part reduction burns; join them back and you have the configuration again.* ***Statement.*** *For every a,*

$$a \;=\; \mathsf{k}a \;\vee\; \partial a \;=\; {\sim}{\sim}a \;\vee\; (a \wedge {\sim}a).$$

Meaning. *The interior $\mathsf{k}a$ is the reversible core — the largest part of a that the closure does not distort — and the boundary ∂a is the exact remainder. The identity says the two together lose nothing: a is recoverable from (core, boundary). This is the content-addressed receipt of a reduction upgraded from a single class* — what was computed — *to a pair* — what was computed, and what was destroyed. *A closure-only calculus can record only the first.* ***Proof.*** *On Ω_3: at $\bot$, $\mathsf{k}\bot \vee \partial\bot = \bot \vee \bot = \bot$; at s, $\mathsf{k}\mathsf{s} \vee \partial\mathsf{s} = \bot \vee \mathsf{s} = \mathsf{s}$; at $\top$, $\mathsf{k}\top \vee \partial\top = \top \vee \bot = \top$. The interior collapses the non-classical middle to $\bot$ and keeps the poles; the boundary is the identity on the middle and $\bot$ on the poles; their join is the identity. Checked in the kernel with no axioms (§8).*

The theorem needs the *interior*, not the closure, and the distinction is not cosmetic. Substitute R for k and evaluate at the middle: $\mathsf{R}\mathsf{s} \vee \partial\mathsf{s} = \top \vee \mathsf{s} = \top \neq \mathsf{s}$. The closure over-completes; only the interior gives back exactly a. This is the mathematical face of why the second negation is load-bearing — the first one is already spoken for by normalization.

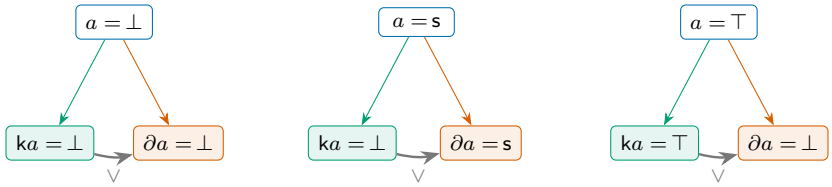


Figure 2: **The fundamental theorem, value by value.** Each configuration a (blue) splits into its interior core $\mathsf{k}a$ (green) and its boundary ∂a (vermil); the join returns a. Only the middle value s has a nonzero boundary — the single non-classical configuration, where all the reverse, the derivative, and the clock draw their content. The poles carry a trivial receipt: nothing is destroyed reducing something already classical.

The Landauer ledger. If the boundary is what reduction destroys, it should have a cost. Assign each configuration the number of bits its boundary carries — zero at the classical poles, one at the non-classical middle — and call it the dissipation $D(a)$ of normalizing a.

Proposition 3.2 (Reversibility is the vanishing of dissipation)**.** *$D(a) = 0$ if and only if $\partial a = \bot$, i.e. if and only if a is classical; and $D(a) > 0$ exactly at the non-classical middle. Normalizing a configuration already at a normal form destroys no information; the receipt is trivial precisely when the boundary is.*

Landauer's principle — erasure has a thermodynamic cost — appears here as an identity, not an analogy: the only irreversible act is erasing the boundary, and its cost is the boundary's bit content. The categorical form of this ledger, a boundary-to-bulk adjunction between an informational and a thermodynamic entropy system, is where the finite statement lifts; that lift is existing substrate we cite rather than reprove.

4 The derivative: a boundary that obeys a product rule

Consider what happens to a boundary when you put two computations side by side. In an interaction net, placing two nets in parallel is a tensor — juxtaposition — and the active pairs of the whole are the active pairs of the parts. The boundary should be additive over this composition. It is, and the additivity is the shadow of a derivation.

Theorem 4.1 (The boundary is a co-Heyting derivation)**.** ***Picture.*** *The boundary of a meet behaves like the derivative of a product: vary one factor, hold the other, and add.* ***Statement.*** *For all a, b,*

$$\partial(a \wedge b) \;=\; (\partial a \wedge b) \;\vee\; (a \wedge \partial b).$$

Meaning. *This is the Leibniz law, with the meet as the product and the join as the sum. An operator that satisfies it is not an arbitrary unary map; it is an* infinitesimal *— a tangent functional, the algebraic seed of d/dt. The boundary differentiates the lattice.* ***Proof.*** *By evaluation over the nine cases of $\Omega_3 \times \Omega_3$; kernel-checked, no axioms.*

The Leibniz law is the *cross-term* product rule — both factors carried, as in $(fg)' = f'g + fg'$. Its additive shadow governs the parallel tensor directly. Model a parallel net as a list of cells and let the boundary content of the whole be the join of the cells' boundaries.

Proposition 4.2 (The boundary distributes over parallel composition)**.** *For net lists N, M juxtaposed by concatenation,*

$$\partial(N \parallel M) \;=\; \partial N \;\vee\; \partial M.$$

The total boundary defect of a parallel composition is the join of the parts' defects — the boundary is conserved additively under composing computations. This is the linear core of the chain rule: to differentiate a composite, differentiate the parts and combine.

One control makes the direction of the structure precise. The boundary is a derivation with respect to the *meet*, not the join: it is not a join-homomorphism. At the poles-and-middle, $\partial(\top \vee \mathsf{s}) = \partial\top = \bot$, whereas $\partial\top \vee \partial\mathsf{s} = \bot \vee \mathsf{s} = \mathsf{s}$. The tensor on which ∂ differentiates must be built from the meet; that is why the parallel composition, and not an arbitrary lattice operation, is the one that carries a chain rule.

5 The clock: a flow whose fixed points are normal forms

A derivation is the infinitesimal generator of a flow. If the boundary is a derivation, it generates a one-parameter dynamics, and we can ask what stands still under it. The answer is exactly what the calculus already calls finished.

Proposition 5.1 (The flow's fixed locus is the eigenform locus)**.** *The closure R is the discrete reduction step; ∂ is its co-Heyting generator. The fixed points of the flow are the boundary's zero locus, which is the closure's fixed locus, which is the set of normal forms:*

$$\partial a = \bot \iff \mathsf{R}a = a.$$

The unique configuration that does not *stand still — the single value the flow moves — is the non-classical middle s, and it is its own boundary: $\partial\mathsf{s} = \mathsf{s}$.*

A stable self-referential configuration — a fixed point of the build-eigenform, in the sense that re-entering it changes nothing — is exactly a configuration the boundary sends to $\bot$. The calculus's ground state, the place where the flow rests, is the boundaryless locus. Time, on this reading, is what the boundary generates, and it stops exactly where the computation does. The middle value that carries the boundary is the same re-entry half-turn that a companion line of this program identifies with the square root of -1; the clock the boundary drives is, in that identification, imaginary. We record the correspondence and do not lean on it here.

6 The same objects on an actual interaction net

The three-element chain is a laboratory. The claim is about reduction, so the objects must live on the configurations of a real net. They do. Take the configuration lattice of a Lafont interaction net with its four natural states — empty, a single active pair, a commutation redex, and the full net — ordered empty $<$ active $<$ commutation $<$ full. The co-Heyting supplement is subtraction from the full net, $\sim a = \text{full} \smallsetminus a$, and the boundary $\partial a = a \wedge \sim a$ is already the object that distinguishes the interior configurations (an active pair, a commutation) from the classical extremes.

Theorem 6.1 (The fundamental theorem on net configurations)**.** ***Picture.*** *The same interior-plus-boundary split, now on the states of a genuine reduction net.* ***Statement.*** *On the four-configuration lattice, with* $\mathsf{k}a = \sim\sim a$*,*

$$\mathsf{k}a \vee \partial a \;=\; a \qquad \textit{for every configuration } a.$$

Meaning. *The interior collapses the two interior configurations — the active pair and the commutation redex, the states that actually reduce — to empty, and keeps the classical extremes; the boundary is the identity on the interior states and empty on the extremes. Their join is the identity. The dual-nucleus decomposition is not an artifact of the toy chain; it is a fact about the states of an interaction net.* ***Proof.*** *By evaluation over the four configurations. Verified independently in Cubical Agda under its* `-safe` *discipline — no postulates, no holes (§8).*

That the same identity holds, proved a second time under a different type theory and on the genuine net-configuration lattice, is the evidence that the construction tracks reduction and not an accident of the three-element model. The interior configurations of Theorem 6.1 — the active pair and the commutation redex — are exactly the states a reducer acts on; the extremes are exactly the states it leaves alone. The boundary is the reducible part of a net, read off algebraically.

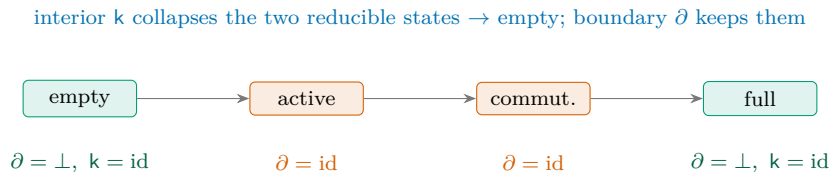


Figure 3: **The net-configuration lattice, with its interior and its boundary.** The four states of a Lafont interaction net form a chain. The two *interior* states (vermil: an active pair, a commutation redex — the states a reducer acts on) carry the boundary; the two classical extremes (green: empty, full) do not. The fundamental theorem $\mathsf{k}a \vee \partial a = a$ holds across the chain (Theorem 6.1), verified independently in Cubical Agda.

7 Idempotent boundary cohomology: projector, reduction, character

A derivation invites a differential, but the boundary itself was the wrong object to force into the simplicial template. The kernel-checked fact is sharper:

$$\partial^2 = \partial.$$

The boundary is an idempotent projector. On the classifier, $\partial(\partial\mathsf{s}) = \partial\mathsf{s} = \mathsf{s}$ is therefore not a failed nilpotence computation; it is the projector's fixed non-classical image.

Proposition 7.1 (The boundary is a projector)**.** *On the classifier, $\partial(\partial a) = \partial a$ for every a. The same finite projector computation is evaluated on the contract corpus consisting of Ω_3, the four-state net-configuration lattice, and the non-chain $\Omega_3 \times \Omega_3$ product.*

The projector has a K-theoretic finite shadow: its rank is its *trace* — the number of non-bottom fixed points of ∂, which equals the number of distinct image cells *because* ∂ is idempotent (the image and the fixed-point set coincide, a theorem consuming $\partial^2 = \partial$). On the contract corpus the ranks are $1, 2, 3$. On the non-chain product all three non-bottom fixed points $(\mathsf{s}, \bot)$, $(\bot, \mathsf{s})$, (s, s) count — the diagonal is a genuine fixed point, so the honest value is 3, not the lattice-atom count 2. Rank is a Murray–von Neumann invariant: any partial isometry of the boundary-generator lists preserves it, the equality *derived* from a bijection rather than carried as data. The three corpus ranks are distinct, so the invariant separates all three carriers.

The differential appears only after coupling the projector to reduction. On a birthday-graded finite carrier, define $d = \mathrm{step} \circ \partial$. The named non-classical active cell moves to a strictly lower-grade residue, so d is not the zero map; and $d^2 = 0$ is a confluence fact — after a projected defect reduces, its next boundary is bottom.

Proposition 7.2 (The reduction homology is concentrated in degree zero)**.** ***Picture.*** *An idempotent's complex leaves no room for higher classes: every boundary is already a cycle, and every cycle is already a boundary.* ***Statement.*** *For the idempotent ∂, the image equals the fixed-point set; hence the reduction complex is acyclic in positive degrees, and the homology is concentrated in degree zero, where H^0 is the fixed-point space — the boundary K_0 — of dimension the rank. A named generator of H^0 on the net carrier is the commutation cell, a non-bottom fixed point.* ***Meaning.*** *An earlier reading recorded a "non-trivial homology of dimension two" by subtracting cardinalities on a set with no linear structure; that is not a homology dimension. The genuine homology is the K-theory, in degree zero, and the positive degrees vanish — which is exactly what confluence buys.* ***Proof.*** $\mathrm{im}\,\partial \subseteq \mathrm{fix}\,\partial$ *consumes idempotency ($\partial(\partial a) = \partial a$);* $\mathrm{fix}\,\partial \subseteq \mathrm{im}\,\partial$ *holds because a fixed point is its own image. Kernel-checked, axiom-clean.*

The degree-zero Chern–Connes character of the idempotent is its trace, and the trace equals the rank (image dimension) *because* ∂ is idempotent — a theorem, not a definitional alias. Since only H^0 survives, the Euler characteristic equals $\dim H^0 = \mathrm{rank}$. The three routes therefore meet at one number, the projector rank ($1, 2, 3$ on the corpus):

$$\mathrm{rank}(\partial) = \chi = \mathrm{ch}_0(\partial),$$

and the character is invariant under the modular automorphism — the product swap $(a, b) \mapsto (b, a)$, which commutes with ∂ and so permutes its fixed points.

Proposition 7.3 (The forward-observe / reverse-prepare lossless primitive)**.** ***Picture.*** *A clean projector can be run both ways without loss: observe a configuration into its settled part, prepare a settled state back into the carrier, and the round trip is the identity.* ***Statement.*** *Because ∂ is idempotent it* splits*: the fixed-point subobject is a retract of the carrier, with a total forward map* observe *(the retraction) and a reverse map* prepare *(the section) satisfying* observe $\circ$ prepare $=$ id *and* prepare $\circ$ observe $= \partial$*. Observation is non-disturbing (*observe $\circ$ observe $=$ observe*), and the whole input is recovered from its interior core and boundary ($a = \mathsf{k}a \vee \partial a$).* ***Meaning.*** *This is the precise, universal form of "translate but stay orthogonal to the input": a forward observation that reads only the invariant and disturbs nothing, and a reverse preparation that reconstructs without loss — a lossless compile/decompile pair, proven for every idempotent, not an analogy. The defining identities are finite and decidable, so a benchmark validates the round-trip fidelity directly (it is* 1 *on every corpus carrier).* ***Proof.*** observe $\circ$ prepare $=$ id *is $\partial x = x$ for a fixed x; the rest is the split. Axiom-free.*

The infinite direction is the periodic-cyclic tower HP^*, and its finite shadow is now in hand: the idempotent induces, on $\mathbb{F}_2$-valued chains, an endomorphism E and its complement whose alternation is a genuine, 2-periodic chain complex — consecutive maps compose to zero because $\partial^2 = \partial$, and the differentials repeat with period two. The whole tower carries a single invariant, the trace = rank; there is no exotic higher structure, which is itself the content.

Proposition 7.4 (Idempotent rigidity: why the tower degenerates)**.** ***Picture.*** *An idempotent cannot be pushed infinitesimally — it is a rigid point, so no differential form is tangent to it, so the tower has nothing above degree zero to see.* ***Statement.*** *For every Leibniz map D (in particular every derivation) on a commutative ring and every idempotent e, $De = 0$.* ***Meaning.*** *The periodic-cyclic tower above degree zero is built from differential forms, i.e. from derivations; by HKR the cyclic tower of a smooth commutative algebra is its de Rham complex. Rigidity says the forms tangent to the idempotent vanish, so the infinite (b, B) tower degenerates onto Hochschild degree zero, where the only class is the trace = rank. The "no exotic higher structure" of the finite shadow is therefore not a gap we declined to fill — it is a rigidity we* prove*, and it settles the whole infinite tower.* ***Proof.*** *From $De = e\cdot De + e\cdot De$ (Leibniz on e^2) multiply by e and use $e^2 = e$: $e\cdot De = e\cdot De + e\cdot De$, so additive cancellation gives $e\cdot De = 0$ and hence $De = 0$. Elementary, self-contained, kernel-clean (no choice); a non-trivial split idempotent $(1, 0) \in A \times B$ witnesses non-vacuity.*

A fully general cyclic-homology library (the arbitrary-algebra (b, B) construction) is not attempted here — the proof assistant's ambient library has none — but the degeneration of *this* tower, which is what the program needed, is settled by Proposition 7.4.

8 Formalization

The mathematics above is machine-checked. The three-element development — the two negations, the boundary, the interior co-nucleus, the fundamental theorem, the Landauer ledger, the Leibniz law and its additive tensor shadow, the flow's fixed locus, the boundary projector, and the finite reduction-homology invariant — is verified in Lean 4 against Mathlib, and every load-bearing statement carries an *empty* axiom footprint: it depends on no axioms at all, not even propositional extensionality. The proofs are finite evaluations, decidable and constructive. An independent re-computation of the same statements, outside the proof assistant, agrees on every case, so the check does not merely confirm the kernel against itself.

The net-configuration development — the interior co-nucleus and the fundamental theorem on the four-state Lafont lattice (Theorem 6.1) — is verified a second time in Cubical Agda under its `-safe` discipline, which forbids postulates and unfilled holes. The two verifications use different type theories and different models of the carrier; their agreement is the paper's principal robustness evidence.

The remaining load-bearing negative is proved, not assumed: the interior and the closure are *not* an adjoint pair (the naive adjunction fails at a single explicit point). The former "boundary is not nilpotent" obstruction has been sharpened rather than hidden: the boundary is a projector, and the finite cohomology lives in the reduction-coupled differential, not in forcing ∂ itself to be square-zero.

Full declaration names, module paths, per-theorem axiom footprints, the independent-oracle entry point, and the verification commands are collected in Appendix A for agents who will extend the work.

9 Realization

The results above are proved in Lean 4. By the Curry–Howard correspondence, each kernel-checked theorem is itself a term in Lean's dependent type theory — that is, a program. Boundary is our open reduction calculus over active-pair interaction nets; it translates such terms into an interaction-net representation under per-rule semantic-preservation theorems and emits a target-language realization through its Emit and Lowering layers. Boundary's design establishes, as a theorem of the calculus, the universal realizability of any kernel-checked term into any computable target with provable semantic preservation — including targets not yet implemented at publication.

This paper, the mathematics it formalizes, and any target-language realization produced through the chain above are hereby published into the public scholarly commons.

10 Forward work: the phases still to land

This is a living document; the construction has room to grow, and the growth is scoped, not gestured. The finite cohomology reframe has landed; the remaining developments are these.

- **The interface cross-term on the net tensor.** Proposition 4.2 proves the boundary additive over disjoint parallel composition. The full product rule — with the interface term that appears when two nets are wired together at a shared port — is the lift of the Leibniz law (Theorem 4.1) from the lattice meet to the genuine monoidal tensor of the net algebra. The classifier-level cross-term already exists; the net-level one is the target.
- **Co-confluence of the interior.** Normalization is confluent modulo net-isomorphism, which is what makes the forward receipt well defined. The reverse receipt — the (core, boundary) pair of Theorem 3.1 — is well defined to the extent that the interior co-reduction converges the same way. Whether anti-normalization is co-confluent modulo net-iso is the well-definedness obligation for the reversible receipt.
- **The periodic-cyclic tower (open).** Proposition 7.1 and Proposition 7.2 land the finite routes: K-theoretic rank, reduction-homology, and degree-zero Chern character. The honest remaining frontier is the infinite periodic cyclic complex HP^*, not another finite rank computation.

Each landed phase adds a section here and a row to the formal record; the paper is meant to be read again as it grows.

A Technical appendix for agents

This appendix is for agents extending the work; it carries the machine-facing detail the body omits.

Lean development. Module `HeytingLean.Logic.BiHeyting.LawvereDefect` (`lean/HeytingLean/Logic/BiHeyting/LawvereDefect.lean`), namespace Ω_3. Every declaration below is verified *axiom-free* (`#print axioms` reports no dependency):

Paper object	Lean declaration
Definition 2.1 interior	`coNucleus`, `coNucleus_idem`
Theorem 3.1	`boundary_fundamental`
Theorem 3.1 antibody	`fundamental_needs_interior`
adjunction refutation	`coNucleus_not_adjoint_R`
Proposition 2.2	`clopen_iff`, `bdry_nonzero_fixed`
Proposition 3.2	`reversible_iff_no_dissipation`, `dissipation_pos_iff_swap`
Theorem 4.1	`meet_tensor_leibniz` (`bdry_leibniz`)
Proposition 4.2	`cellBdry_tensor`
Proposition 4.2 control	`bdry_not_join_hom`
Proposition 5.1	`flow_fixed_iff_eigenform`, `time_generator_unique`
non-nilpotence control	`boundary_not_a_differential`, `nilpotent_off_swap`
Proposition 7.1	`bdry_idempotent`, `trace_eq_imageRank`, `boundaryRank_nonchain`, `rank_mvn_invariant`, `rank`
Proposition 7.2	`image_is_cycle`, `cycle_is_boundary`, `h0_nontrivial`, `d_squared_zero`
glue	`chern_eq_image_prod`, `chern_modular_invariant`, `rank_euler_chern_coincide`
Proposition 7.3	`roundtrip_lossless`, `projector_recovered`, `full_reconstruction`
tower	`tower_composite_vanishes`, `tower_periodic`, `chernTrace`
Proposition 7.4	`idempotent_derivation_rigid`, `tower_degenerates_to_trace`, `nontrivial_idempotent_exis`

Boundary cohomology development. Modules `HeytingLean.Logic.BiHeyting.BoundaryKTheory` and `HeytingLean.Boundary.Cohomology.{GradedCarrier,ReductionDifferential,ReductionHomology,Cher` carry the projector rank (the trace of the idempotent), finite Murray–von Neumann invariance derived from a list bijection, the reduction-coupled differential, the degree-zero homology H^0 with its named generator, the degree-zero Chern character, and the glue theorem. The rank is the count of non-bottom fixed points of the boundary, computed on explicit finite carriers including the full nine-element non-chain $\Omega_3 \times \Omega_3$ product (honest value 3); the homology is concentrated in degree zero, where $\dim H^0$ equals the rank, the positive degrees being acyclic because the boundary's image coincides with its fixed-point set.

Agda development. Theorem 6.1 on the net-configuration lattice: `corpus/cubical_agda/PolygraphBoundary/` — `coNucleus`, `coNucleus-idem`, `boundary-fundamental`, `closure-decoy-yields-full`. Verified: `agda -safe -cubical` (exit 0, no postulates, no holes). The reduction differential mirror lives at `corpus/cubical_agda/PolygraphBoundary/NetReductionDifferential.agda`.

Independent oracle. `scripts/check_dual_nucleus.py` recomputes the original dual-nucleus statements by finite enumeration, without Lean. The cohomology extension has its own corpus battery under `tests/external_oracles/biheyting_boundary_cohomology_20260701/`; `phase_1_oracle.py` through `phase_6_oracle.py` and `closeout_audit.py` recompute the projector, rank, differential, homology, Chern character, glue, and proof-term consumption checks.

Catalog. Registry object `repo_native_objects.biheyting_dual_nucleus_boundary` (query `mathematical_object_registry.py -get biheyting_dual_nucleus_boundary`); semantic overlay `semantic_overlay/declarations/logic_bi_heyting_lawvere_defect.json`; conjecture `biheyting_dual_nucleus_boundary_20260630`. The cohomology extension is indexed as `repo_native_objects.biheyting_boundary_cohomology`.

Provenance. Landed on `origin/master`: `35a6460d5c` (Lean phases), `96c99ee4c4` (Agda netconfig implementation), `c82145473f` (registry catalog).

Acknowledgment. We humbly thank the collective intelligence of humanity for providing the technology and culture we cherish. We do our best to properly reference the authors of the works utilized herein, though we may occasionally fall short. Our formalization acts as a reciprocal validation—confirming the structural integrity of their original insights while securing the foundation upon which we build. In truth, all creative work is derivative; we stand on the shoulders of those who came before, and our contributions are simply the next link in an unbroken chain of human ingenuity.

The Institute for Applied Ontological Mathematics. A living document; extended as phases land. Correspondence and the growing formal record travel with the repository, not with individuals.